Maths with Melissa

Midpoint Proof

Introduction

In geometry, the midpoint of a line segment is the point that divides the segment into two equal parts. If you have two points, $A(x_1, y_1)$ and $B(x_2, y_2)$, the midpoint M is given by:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Below is a step-by-step proof of why this formula gives the correct midpoint.

Proof: Midpoint Formula

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in the Cartesian plane. We want to find the point $M(x_m, y_m)$ that is exactly halfway between A and B.

Step 1: Find the change in x and y

The horizontal distance between A and B is: $\Delta x = x_2 - x_1$

The vertical distance between A and B is: $\Delta y = y_2 - y_1$

Step 2: Find the halfway point

To find the midpoint, take half of the horizontal and vertical distances and add them to the coordinates of A:

$x_m = x_1 + \frac{\Delta x}{2} = x_1 + \frac{x_2 - x_1}{2}$ $y_m = y_1 + \frac{\Delta y}{2} = y_1 + \frac{y_2 - y_1}{2}$

Step 3: Simplify the expressions

$x_m = x_1 + \frac{x_2 - x_1}{2} = \frac{2x_1 + x_2 - x_1}{2} = \frac{x_1 + x_2}{2}$ $y_m = y_1 + \frac{y_2 - y_1}{2} = \frac{2y_1 + y_2 - y_1}{2} = \frac{y_1 + y_2}{2}$

Step 4: State the result

Therefore, the midpoint M has coordinates: $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Example

Let $A(2,3)$ and $B(6,11)$. Find the midpoint M.

$$M = \left(\frac{2+6}{2}, \frac{3+11}{2}\right) = (4,7)$$

So, the midpoint is (4,7).

> **Important**: The midpoint formula works in any dimension. For three dimensions, simply add the z-coordinates and divide by 2 as well.

Midpoint Proof Questions

1. Given points $A(1,5)$ and $B(7,9)$, use the proof steps to show that the midpoint M is $(4,7)$. Write out each step explicitly.
2. Explain why taking half the horizontal and vertical distances and adding them to the coordinates of A guarantees that M is exactly halfway between A and B.
3. If the midpoint M of $A(x_1, y_1)$ and $B(x_2, y_2)$ is (m_x, m_y), prove that the distance from A to M is equal to the distance from M to B using the distance formula.
4. Given $A(-3,2)$ and $B(5,-6)$, show how the midpoint formula is derived from the general idea of averaging the coordinates, and verify the result using the proof steps.
5. In your own words, summarise the logical flow of the midpoint proof and explain why each step is necessary in establishing the formula.

Answer Key

1. **Step-by-step for A(1,5) and B(7,9):**
   - $\Delta x = 7 - 1 = 6$, $\Delta y = 9 - 5 = 4$
   - Halfway: $x_m = 1 + 6/2 = 1 + 3 = 4$, $y_m = 5 + 4/2 = 5 + 2 = 7$
   - So, midpoint $M = (4,7)$.

2. **Explanation:**
   - Because half the distance from A to B in both x and y directions places M equidistant from both endpoints, ensuring M is exactly halfway.

3. **Proof using distance formula:**
   - Distance $AM = MB = \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2}$, which confirms both segments are equal.

4. **For A(-3,2), B(5,-6):**
   - Average x: $(-3 + 5)/2 = 1$, average y: $(2 + -6)/2 = -2$
   - Proof steps: $\Delta x = 8$, $x_m = -3 + 8/2 = 1$; $\Delta y = -8$, $y_m = 2 + (-8)/2 = -2$
   - Result matches: $(1, -2)$.

5. **Summary:**
   - The proof shows the midpoint is found by averaging coordinates, justified by halving the distance in each direction and ensuring the point is equidistant from both endpoints.